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A method for calculating the stress–strain state in the general boundary-value problem of metal forming—part 1

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Abstract

To simulate metal-forming processes, one has to calculate the stress–strain state of the metal, i.e. to solve the relevant boundary-value problems. Progress in the theory of plasticity in that respect is well known, for example, via the slip-line method, the finite element method, etc.), yet many unsolved problems remain. It is well known that the slip-line method is scanty. In our opinion the finite element method has an essential drawback. (No one is against the idea of the discretization of the body being deformed and the approximation of the fields of mechanical variables.) The results of calculation of the stress state by the FEM do not satisfy Newtonian mechanics equations (these equations are said to be "softened", i.e, satisfied approximately) and stress fields can be considered "poor" for solution of the subsequent fracture problem. We believe that it is preferable to construct an approximate solution by the FEM and "soften" the constitutive relations (not Newtonian mechanics equations), especially as, in any event, they describe the rheology of actual deformable materials only approximately. We seem to have succeeded in finding the solution technique.

Here we present some new results for solving rather general boundary-value problems which can be characterized by the following: the anisotropy of the materials handled; the heredity of their properties and compressibility; finite deformations; non-isothermal flow; rapid flow, with inertial forces; a non-stationary state; movable boundaries; alternating and non-classical boundary conditions, etc.

Solution by the method proposed can be made in two stages: (1) integration in space with fixed time, with an accuracy in respect of some parameters; (2) integration in time of certain ordinary differential equations for these parameters.

In the first stage the method is based on the principle of virtual velocities and stresses. It is proved that a solution does exist and that it is the only possible one. The approximate solution "softens" (approximately satisfies) the constitutive relations, all the rest of the equations of mechanics being satisfied precisely. The method is illustrated by some test examples.

1. Introduction

Recent papers and books published (see e.g. Chenot et al., 1995; Zhong, 1993; Gerhardt, 1989; Pozdeev et al., 1986) speak of the state of the art in the calculation of the stress–strain state in

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metal forming with application of the finite element method (FEM). Naturally, the calculation is approximate, and, notwithstanding the progress in computer hardware, the matter of topical importance is to find a better method of approximate solution.

To calculate the stress–strain state in metal forming and, on the whole, in deformation mechanics is to solve the boundary value problems of continuum mechanics. Continuum mechanics equations may be schematically divided into three types: kinematic equations, dynamic equations and constitutive relations. The characteristic feature of the majority of the above mentioned works is the fact that they satisfy the kinematic equations exactly, satisfy the constitutive relations, but, strictly speaking, they do not satisfy the dynamic equations. In fact, the solution is constructed in velocities and displacements, e.g., by Lagrange's, Jourdain's, Markov's principles or Galerkin's method. Then, by the flow kinematics (in the case of Markov's principle, by kinematics and mean normal stress), stress tensor fields are found by means of constitutive relations.

Since, for example, direct variational methods are applied for the solution, the stress fields obtained do not satisfy the dynamic equations, namely, the differential equations of balance (or motion, if the flow has mass inertial forces), the stresses do not accurately satisfy the boundary conditions in stresses (therefore one can speak of "softer" satisfiability). Certainly, as the number of variation parameters grows, and/or if more suitable coordinate functions are used, the discrepancy in the satisfaction must decrease. Even the solution of very complicated problems by the FEM (as in Chenot et al., 1995; Zhong, 1993; Gerhardt, 1989; Pozdeev et al., 1986) seem very similar. The man has accumulated a certain experience in body forming, and the visualization of the kinematic solution results (displacements, velocity fields and even strain distribution) creates the impression of safety. However, the man has little physical notion of stress fields, and paper authors seldom bring their results to the analysis of the stress state obtained. We do not state that this method of approximate solution is worse than the one described in the present paper, but we are of the opinion that the latter method deserves consideration.

By the alternative method, we seek the solution in the form of kinematic fields (of velocities, displacements, etc.) satisfying all the kinematic equations and in the form of stress fields satisfying all the dynamic equations. Since the solution is still approximate, the "softening" falls on the constitutive relations (as the accuracy of the approximate solution grows, the discrepancy in the satisfiability decreases). It should be noted that the constitutive relations are always approximate and found from experiments where there are experimental errors, therefore this way of "softening" seems more preferable to the author of this paper.

The idea of simultaneous varying the stress and strain states with "softening" only the constitutive relations is not new. It was proposed independently by Kolarov et al. (1979) and Kolmogorov (1967) and advanced considerably in Kolmogorov (1970), Unksov et al. (1983), Kolmogorov (1986), Kolmogorov and Lapovok (1992), Unksov et al. (1992) and Fedotov (1990).

2. Correct formulation of the general boundary-value problem

The boundary-value problem consists in the integration of the system of equations of continuum mechanics with respect to the variables describing the kinematic flow and the stress state for certain boundary and initial conditions. Some of these equations (the so-called constitutive relations) and the boundary conditions are formulated from experiments for the specific class of problems prescribed, particularly, those of metal-forming mechanics; any formulation must however be correct. A correct formulation is as follows.

Let a material body with volume V undergo finite plastic deformation. Let the constitutive relations be given $\forall M \in V$. They can be given in any form (e.g., in volume V_p (part of V) they describe plastic deformation, and in volume V_e, the rest of the volume which is elastic, etc.). We assume that, under conditions of developed forming, the material being formed possesses rheonom properties, the constitutive relations of which are known functionals of the history of deformation development with time, temperature θ and density ρ, etc. However, at every fixed moment of time t, including the one under study, they turn into some known tensor functions, together with their inverse functions

$$s^{ij} = s^{ij}(e_{kl}), \quad e_{ij} = e_{ij}(s^{kl}); \tag{1}$$

$$\sigma = \sigma(\xi), \quad \xi = \xi(\sigma). \tag{2}$$

Here, s^{ij} and e_{ij} are the components of the stress deviators and the deformation velocities; σ and ξ are mean normal stress and the rate of relative volume change, respectively; e_{kl} and s^{kl} form a set of deviator components which appear in the functions (1) as arguments. Among the arguments in the functions (1) and (2) there may be any characteristics of the stress and strain states (e.g., derivatives $\dot{s}^{ij}$, $\dot{\sigma}$, etc.), but in (1) and (2) there are arguments that are principal ones for the following reasoning. The coordinates adopted are Lagrangian ones. Let the functions (1) and (2) satisfy the conditions

$$\partial s^{ij}/\partial e_{kl}\,|_{\substack{i=k\\j=l}} > 0 \tag{3}$$

$$\partial \sigma/\partial \xi > 0. \tag{4}$$

So, constitutive relations at any fixed instant of time t—functions (1) and (2)—must be differentiable with respect to the mentioned arguments and have inverse functions; for the functions, conditions (3) and (4) must be satisfied because they express the known properties of metal viscosity.

Let the solution sought be in the velocity fields v_i, which is continuous in the coordinates of V, and in the V surface stress fields, $f^i = \sigma^{ij} n_j$. Here σ^{ij} are the components of the stress tensor; $\mathbf{n}$ is the unit normal to the surface.

Suppose the body undergoing to deformation, have a volume V, bounded by a surface S which consists of the parts S_v, S_f and S_S; the boundary conditions on them are assumed to be as follows:

$$\forall M \in S_v, \quad v_i = v_i^*; \tag{5}$$

$$\forall M \in S_f, \quad f_i = \sigma^{ij} n_j = f^i{}_*; \tag{6}$$

$$\forall M \in S_S, \quad v_{\mathrm{v}} = v_{\mathrm{v}}^*, \quad \mathbf{f}_\tau = f_\tau(f_{\mathrm{v}}, v_S)\mathbf{i}. \tag{7}$$

Here, $v_i^*, f^i{}_*, v_{\mathrm{v}}^*$ are given functions (everywhere marked with an asterisk) of coordinates on the surface S and time, t; $\mathbf{v}_S$ is the tool slip vector and $\mathbf{i} = \mathbf{v}_S/v_S$; v_{v}^* and f_{v} are the normal (to the surface S) components of vectors; $f_\tau = f_\tau(f_{\mathrm{v}}, v_S)$ is the known friction law. The friction law may be the functional of the particle's trajectory on the surface S_S, but at a fixed moment in time t, it must be represented by a known function, resolvable with respect to $v_S[v_S(f_v, f_\tau)]$, and it must satisfy the following condition:

$$\partial f_\tau / \partial v_S > 0. \tag{8}$$

In the function $f_\tau = f_\tau(f_v, v_S)$ any other quantities (e.g., $u_S = \int_0^t v_S \, d\tau$, the displacement of a particle on S_S) can be present as arguments. Conditions for the continuity of v_i and f^i must be satisfied on the boundaries (lines that, generally speaking, are not known) between S_v, S_f and S_S. The relation (8) refers to the viscous properties of the metal and the lubricant.

Of course, boundary conditions should be given for the temperature part of the problem, but we do not deal with the temperature part in the present paper.

Finally, suppose that $\forall M \in V$ distributed mass forces g_i^* are given.

Suppose, to integrate in time, for every material particle $M \in V$, the following initial conditions being given (at $t = 0$):

$$\forall M \in V, \quad v_i = v_i^0, \quad \sigma^{ij} = \sigma_0^{ij}, \quad \rho = \rho_0. \tag{9}$$

On the right, marked by zero, are the known functions of the coordinates.

Thus, we have formulated the boundary-value problem for the mechanics of a body undergoing deformation.

Consider the solution for this problem.

3. The principle of virtual velocities and stresses, and the integration of the boundary-value problem in space

Consider an unspecified but fixed instant of time t. The integration of the boundary-value problem at issue in space can be replaced by the equivalent task of solving the following variational equation for the principle of virtual velocities and stresses:

$$\delta I = 0, \tag{10}$$

where

$$I = \int_V \left[\int_0^{e'_{ij}} s^{ij}(e) \, de + \int_0^{s^{ij\prime}} e_{ij}(s) \, ds + \int_0^{\xi'} \sigma(\xi) \, d\xi + \int_0^{\sigma'} \xi(\sigma) \, d\sigma + \rho(w^i - g_i^*) v'_i \right] dV$$
$$- \int_{S_f} f^i{}_* v'_i \, dS - \int_{S_v} f^{i\prime} v_i^* \, dS - \int_{S_S} \left[f^{i\prime} v_i^* - \int_0^{v'_{si}} f_\tau^i(v) \, dv - \int_0^{f^{i\prime}} v_{si}(f) \, df \right] ds.$$

Variation proceeds isochronously only with respect to virtual quantities that are marked in (10) by a prime. The summation is made over the indices i and j which appear in the upper limits of the integrals and in the expressions under the integral sign. Naturally, the constitutive relations must be such that the functional I in (10) (curly brackets) was differentiable. The virtual v'_i must satisfy (side by side with the continuity in V and on S) the following conditions:

$$\forall M \in V, \quad d\rho/dt + \rho \operatorname{div} \mathbf{v}' = 0;$$

$$\forall M \in S_v, \quad v_i' = v_i^*;$$
$$\forall M \in S_S, \quad v_v' = v_v^*. \tag{11}$$

The virtual $\sigma^{ij'}$ must satisfy (side-by-side with the continuity of the surface stresses $f^{i'} = \sigma^{ij'} n_j$ in V and S) the following conditions:

$$\forall M \in V, \quad \nabla_i \sigma^{ij'} + \rho(g^i{}_* - w^j) = 0, \quad \sigma^{ij'} = \sigma^{ji'};$$
$$\forall M \in S_f, \quad \sigma^{ij'} n_j = f^i{}_*. \tag{12}$$

Here w^j refers to the acceleration of material particles. Note that the virtual $\sigma^{ij'}$ and v_i' satisfy all the equations of continuum mechanics (which are linear in this case, and which simplifies the practical application of the principle of virtual velocities and stresses), except for the constitutive relations.

The functional I, calculated for any virtual stress–strain state, even entirely different from the actual one, but with other conditions equal (i.e. invariable quantities), is not negative and becomes zero on achieving the absolute minimum at the actual state, which is the solution of the boundary-value problem at issue in space at a fixed time. The quantities I, calculated for some virtual state described by the fields v_i' and $\sigma^{ij'}$, expresses a discrepancy in their satisfying the constitutive relations.

The solution of the variational problem (10) does exist and is unique. In view of equivalency there exists (and it is unique) a solution (named "actual fields" v_i and σ^{ij}) for the following problem:

$$\nabla_i [s^{ij}(e_{kl}) + \sigma(\xi) g^{ij}] = \rho(w^j - g^j{}_*);$$
$$e_{ij}(s^{kl}) + \xi(\sigma) g^{ij}/3 = (\nabla_i v_j + \nabla_j v_i)/2 \tag{13}$$

with the boundary conditions (5)–(7). Here ∇_i is a covariant derivative operator; g^{ij} are metric tensor components; $e_{kl} = (\nabla_k v_l + \nabla_l v_k)/2$; $\xi = \xi_{kl} g^{kl}$; $s^{kl} = \sigma^{kl} - \sigma g^{kl}$ and $\sigma = \sigma^{kl} g_{kl}/3$.

4. An approximate solution for the general boundary-value problem as a whole

The approximate solution at an unspecified instant of time t will be sought by using the principle of virtual velocities and stresses in the form

$$v_i' = \sum_{k=1}^{n} a_{ki} v_{ki}(x);$$

$$\sigma^{ij\prime} = \sum_{k=1}^{m} b_k^{ij} \sigma_k^{ij}(x). \tag{14}$$

Here, x are Lagrangian coordinates; a_{ki} and b_k^{ij} are variable coefficients (at fixed t and, generally speaking, functions of time); $v_{ki}(x)$ and $\sigma_k^{ij}(x)$ are known suitable functions of coordinates (in the right hand part there is no summation over the repetitive indices i, j). The suitable functions are selected so that v_i' and $\sigma^{ij\prime}$ are virtual.

The variation eqn (10) will turn into two groups of equations, thus

$$\int_V \left[s^{ij}(e') \frac{\partial e'_{ij}}{\partial a_{ki}} + \sigma(\xi') \frac{\partial \xi'}{\partial a_{ki}} + \rho(w^i - g^i{}_*) \frac{\partial v'_i}{\partial a_{ki}} \right] \mathrm{d}V$$

$$- \int_{S_f} f^i{}_* \frac{\partial v'_i}{\partial a_{ki}} \mathrm{d}S + \int_{S_S} f_{\tau i}(v'_S) \frac{\partial v'_{si}}{\partial a_{ki}} \mathrm{d}S = 0, \quad k = 1, \ldots, n;$$

$$\int_V \left[e_{ij}(s) \frac{\partial s^{ij'}}{\partial b^{ij}_k} + \xi(\sigma') \frac{\partial \sigma'}{\partial b^{ij}_k} \right] \mathrm{d}V - \int_{S_V} v^*_i \frac{\partial f^{i'}}{\partial b^{ij}_k} \mathrm{d}S - \int_{S_S} \left[v^*_i \frac{\partial f^{i'}}{\partial b^{ij}_k} - v_i(f_\tau) \frac{\partial f^{i\,\prime}_\tau}{\partial b^{ij}_k} \right] \mathrm{d}S = 0. \quad (15)$$

Here, indices i and ij, appearing in the denominator of the partial derivatives do not participate in the summation.

Note that $\boldsymbol{\omega} = \partial \mathbf{v}/\partial t$, $\mathbf{v} = \partial \mathbf{x}/\partial t$ and $\rho = \rho_0 \det \|\partial x^{i0}/\partial y^j\| / \det \|\partial x^k/\partial y^l\|$, where ρ_0 and ρ are the initial and current density of the material, respectively, x^{i0} and x^k are initial and current Eulerian coordinates of the particle, y^j and y^l are their Lagrangian coordinates. Thus, relations (15) mean that the evolutionary problem concerning the change in the stress state of the body and its kinematics is now reduced to the integration of the following two groups of ordinary differential equations (written schematically):

$$\dot{a} = F_1(a, b, \dot{b});$$

$$F_2(a, b, \dot{b}) = 0 \quad (16)$$

with the initial conditions (9). The question of the existence and uniqueness of the solution are taken to be settled mathematically. Here, in (16), it is assumed as an example, that the constitutive relations connect deformation velocities and stress velocities. The result of the solution of the system (16) approximately solves the problem as a whole, thus, (14) will now have the following form:

$$v_i = \sum_{k=1}^{n} a_{ki}(t) v_{ki}(x);$$

$$\sigma^{ij} = \sum_{k=1}^{m} b^{ij}_k(t) \sigma^{ij}_k(x). \quad (17)$$

An approximate solution by the method described will now be presented.

5. Some particular cases and exceptions to the rules in sections 3 and 4

The material treated is often assumed to be isotropic, and the stress and velocity deviators—similar. In this case constitutive relations (1), all in all 10, including inverse functions, are written through invariants only as two functions:

$$T = T(H); \quad H = H(T). \quad (18)$$

Here T is the intensity of the tangential stresses and H is the intensity of shear strain velocities. Then, in eqn (10), the first ten items should be substituted for as follows:

$$\int_0^{e'_{ij}} s^{ij}(e)\,\mathrm{d}e + \int_0^{s^{ij'}} e_{ij}(s)\,\mathrm{d}s = \int_0^{H'} T(\eta)\,\mathrm{d}\eta + \int_0^{T'} H(\tau)\,\mathrm{d}\tau. \tag{19}$$

Often the material does not display viscous properties, in which case the constitutive relations (2), (18) and the friction law in (7) at a fixed instant of time t will certainly have no ξ, H or v_S among the arguments and, consequently, there will be no inverse functions. The variational eqn (10) will be simpler in this case—

$$\delta\left\{\int_V [TH' + \sigma\xi' + \rho(w^i - g^i{}_*)v'_i]\,\mathrm{d}V - \int_{S_f} f^i{}_* v'_i\,\mathrm{d}S' - \int_{S_v} f^{i'} v_i^*\,\mathrm{d}S - \int_{S_S} (f^{i'} v_i^* - f^i_\tau v'_{si})\,\mathrm{d}S\right\} = 0; \tag{20}$$

the following conditions are however added to conditions (12) imposed on the virtual $\sigma^{ij\prime}$:

$$\forall M \in V, \quad T' = T, \quad \sigma' = \sigma;$$

$$\forall M \in S_S, \quad f^{i\,\prime}_\tau = f^i_\tau. \tag{21}$$

Here the right hand parts are stresses arrived at by the instant of time t.

One more simplification pertains to the incompressibility of the material. In this case the second item in (20) will be absent, and the first condition in (11) will assume the form

$$\forall M \in V, \quad \xi' = 0. \tag{22}$$

6. On the fragmentation of the bodies under deformation

In the above-described solution of the boundary-value problem it was assumed (as usual) that the material volume V remains continuous during the deformation, that it does not become divided into parts, and that no macroscopic holes or macro-cracks appear in it, i.e. there is no macroscopic fragmentation of the body under deformation. The considerations and the solution of the boundary-value problem were valid (in as far as continuum mechanics is valid) until the beginning of the macroscopic fragmentation, i.e. the loss of continuity (at instant t_p). At the same time this instant can be considered as the beginning of a new stage, i.e. a new solution for a new boundary-value problem, because, on the new surfaces created, there appear additional boundary conditions, which require a new statement of the boundary-value problem. The second stage will continue until further new surfaces appear, and so on.

The instant t_p at which the fragmentation starts and the instants of further macro-discontinuities can be determined by means of fracture theory (see e.g. Unksov et al., 1983), supplied with some new statements. According to this theory, in every material particle of the body under deformation, accumulation of damage ψ takes place. By the instant of time t, $\psi(t)$ is calculated based on kinematic relations. For that purpose, firstly, we solve the appropriate boundary-value problem. Secondly, we specify (or find from special experiments) plastic characteristics of the body under deformation. Damage ψ is calculated for every material particle. For that purpose, on the trajectory of its motion, we specify separate sections of monotonic deformation. Within the section, the components of the particle deformation velocity tensor do not change the sign. We indicate $t_1, t_2, \ldots, t_{n-1}$—the instants at which the tensor component changes sign (transition through zero of at least one component). At the first section ($t_0 \leqslant t < t_1$) damage is determined as

$$\psi(t) = \psi_1(t),$$

$$\frac{d\psi_1}{dt} = \frac{H(t)}{\lambda_p[k_1(t), k_2(t)]}, \quad \psi_1(t_0) = 0;$$

at the second section $t_1 \leqslant t < t_2$,

$$\psi(t) = [\psi_1(t_1)]^{\alpha_1} + [\psi_2(t)]^{\alpha_2},$$

$$\frac{d\psi_2}{dt} = \frac{H(t)}{\lambda_p[k_1(t), k_2(t)]}, \quad \psi_2(t_1) = 0;$$

at the n-th section $t_{n-1} \leqslant t < t$,

$$\psi(t) = \sum_{i=1}^{n} \psi_i^{\alpha_1},$$

$$\frac{d\psi_n}{dt} = \frac{H(t)}{\lambda_p[k_1(t), k_2(t)]}, \quad \psi_n(t_{n-1}) = 0.$$

Firstly, here we have the results of solving the boundary-value problem: $H = H(t)$ is the intensity of shear deformation velocity; $k_1 = k_1(t)$ and $k_2 = k_2(t)$ are dimensionless independent invariants of the stress tensor [$k_1 = \sigma/T$, $k_2 = 2(\sigma_{22} - \sigma_{33})/(\sigma_{11} - \sigma_{33}) - 1$, where σ is the mean normal stress and T is the intensity of the tangential stresses; $\sigma_{11} \geqslant \sigma_{22} \geqslant \sigma_{33}$ are principal normal stresses]. Secondly, we have the plastic characteristics of the body under deformation, found from experiments: $\lambda_p = \lambda_p(k_1, k_2)$ refers to plasticity values and $\alpha_i = \alpha_i(\bar{k}_1, \bar{k}_2)$ are the values of the function $\alpha = \alpha(k_1, k_2)$ at the i-th section of the monotonic deformation.

By the instant of the fracture ($t = t_p$)

$$\psi(t) = \psi(t_p) = 1, \tag{23}$$

and the body is saturated with microdamage (which, however, does not appear when solving the boundary-value problem), the material becomes embrittled and is about to form a macro-crack (body fragmentation start). Condition (23) marks the end of solving the boundary-value problem within the accepted statement and the beginning of a new stage, i.e., new solution. How can we find t_p, the macro-break spot and formulate the boundary conditions on the new surface?

The calculation of damage by the given algorithm proceeds after integration of ordinary differential equations in time, allowing one (as is described above) to obtain an approximate solution for the boundary-value problem. At every instant of time t, we can solve the problem of seeking the x coordinates of the points in the body volume V, at which the damage is maximum

$$\max_{x \in V} [\psi(t)]. \tag{24}$$

The instant of time $t = t_p$ will be determined when the maximum value of ψ, according to (24), achieves unity, i.e.

$$\max_{x \in V} [\psi(t_p)] = 1. \tag{25}$$

Simultaneously, the point (or points) can be found where the macro-crack appears.

How will the surface of the macro-crack be oriented? We can assume that, if plastic deformation precedes fracture, then, at the instant $t = t_p$, the crack will be oriented along the spots of maximum tangential stresses. The crack will have finite dimensions owing to the continuous change of ψ in the volume V. The dimensions can be calculated from solving a new boundary-value problem and calculating the stress–strain state around the crack.

At the spots of maximum tangential stress, the latter and the normal ones will be as follows (respectively):

$$\tau_n = \tfrac{1}{2}(\sigma_{11} - \sigma_{33});$$
$$\sigma_n = \tfrac{1}{2}(\sigma_{11} + \sigma_{33}). \tag{26}$$

If, at $t = t_p$, in the point with $\psi = \psi_{\max} - 1$, $\sigma_n \geqslant 0$, then there appears a crack with the "edges" free from surface stresses, i.e., $\tau_n = \sigma_n = 0$. On the edges of the crack evolved, there arises impact off-loading to the value of

$$\Delta\tau_n = \tau_n, \quad \Delta\sigma_n = \sigma_n. \tag{27}$$

If, at $t = t_p$, in the point with $\psi = \psi_{\max} = 1$, we have $\sigma_n < 0$, then there appears a crack on split with the edges not free from surface stresses. On the edges there occurs an impact unloading of the value of

$$\Delta\tau_n |\tau_n| - \mu|\sigma_n|, \tag{28}$$

if it is supposed that friction between the split crack banks is according to Coulomb (μ is friction coefficient). In the further solution the split crack banks should be viewed as surfaces with sliding friction.

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